Double integrals over rectangles

Recall:

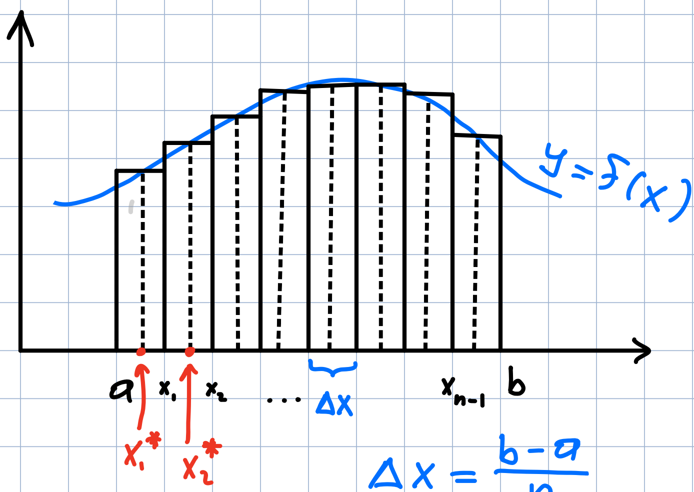
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f(x_i^*)\,\Delta x}_{\text{area of a rectangle}}$$

x_i^* — sample point in $[x_{i-1}, x_i]$

For $f \geq 0$

$\downarrow$

$=$ area of the region $\left\{(x,y) \,\middle|\, \begin{matrix} a \leq x \leq b \\ 0 \leq y \leq f(x) \end{matrix}\right\}$



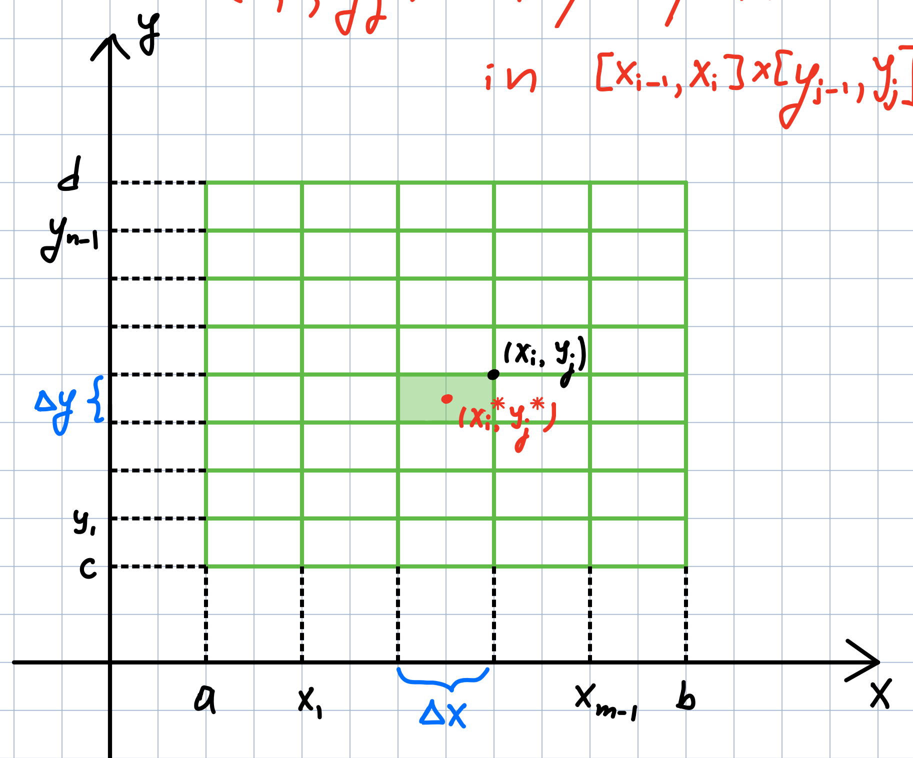
$$\Delta x = \frac{b-a}{n}$$

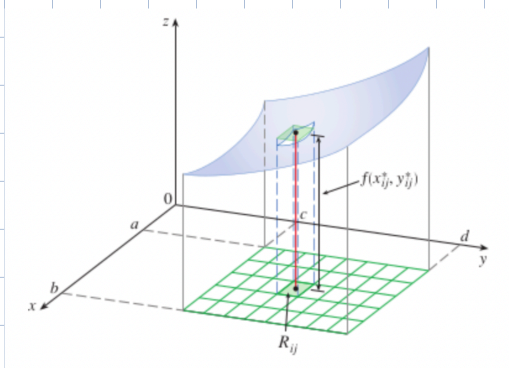
$R = \left\{(x,y) \,\middle|\, \begin{matrix} a \leq x \leq b \\ c \leq y \leq d \end{matrix}\right\}$ – rectangle

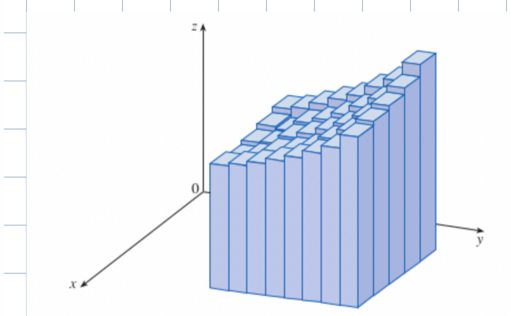
$f(x,y)$ – function

$$\iint_R f(x,y)\,dA = \lim_{\substack{m\to\infty \\ n\to\infty}} \sum_{i,j} \underbrace{f(x_i^*, y_j^*)\,\underbrace{\Delta A}_{\Delta x \cdot \Delta y}}_{\text{volume of a column}}$$

(x_i^*, y_j^*) – sample point in $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$





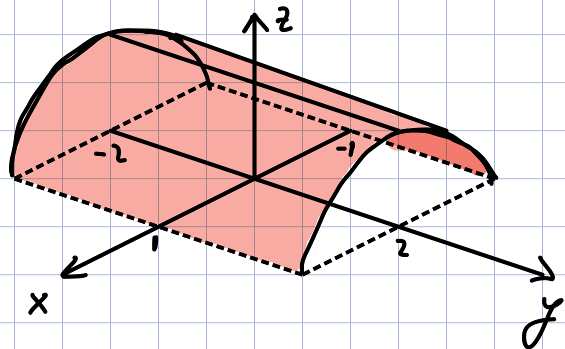


If $f \geq 0$, volume V of a solid that lies above the rectangle R and below the surface $z = f(x,y)$ is:

$$\boxed{V = \iint_R f(x,y)\,dA}$$

Ex: $R = \{(x,y) \mid -1 \leq x \leq 1, -2 \leq y \leq 2\}$, find $\iint_R \underbrace{\sqrt{1-x^2}}_{f(x,y)}\,dA$

Sol:



solid = half-cylinder,

$$V = \frac{1}{2}\pi r^2 L = \frac{1}{2}\pi \cdot 1^2 \cdot 4 = 2\pi$$

Iterated integrals:

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \underbrace{\left(\int_c^d f(x,y)\,dy\right)}_{\substack{A(x)\text{ - partial integration of } f \\ \text{w.r.t. } y\text{, regarding } x \text{ as a const.}}} dx$$

Rmk: Similarly,

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx\right) dy$$

Ex: Find a) $\int_0^3 \int_1^2 x^2 y\,dy\,dx$ b) $\int_1^2 \int_0^3 x^2 y\,dx\,dy$

Sol: (a) $\int_0^3 \underbrace{\left(\int_1^2 x^2 y\,dy\right)}_{x^2 \frac{y^2}{2}\Big|_{y=1}^{y=2} = \frac{3}{2}x^2} dx = \int_0^3 \frac{3}{2}x^2\,dx = \frac{3}{2}\cdot\frac{x^3}{3}\Big|_0^3 = \frac{27}{2}$

(b) $\int_1^2 \underbrace{\left(\int_0^3 x^2 y\,dx\right)}_{y\frac{x^3}{3}\Big|_{x=3}^{0} = 9y} dy = \int_1^2 9y\,dy = 9\frac{y^2}{2}\Big|_1^2 = \frac{9}{2}(4-1) = \frac{27}{2}$

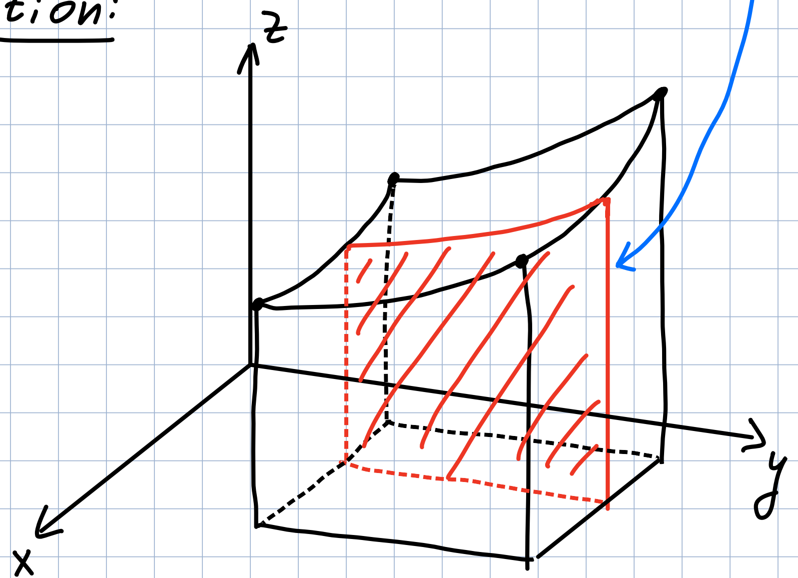
Fubini's thm: if $f(x,y)$ continuous on a rectangle $R = \{(x,y) \mid {}^{a \le x \le b}_{c \le y \le d}\}$,

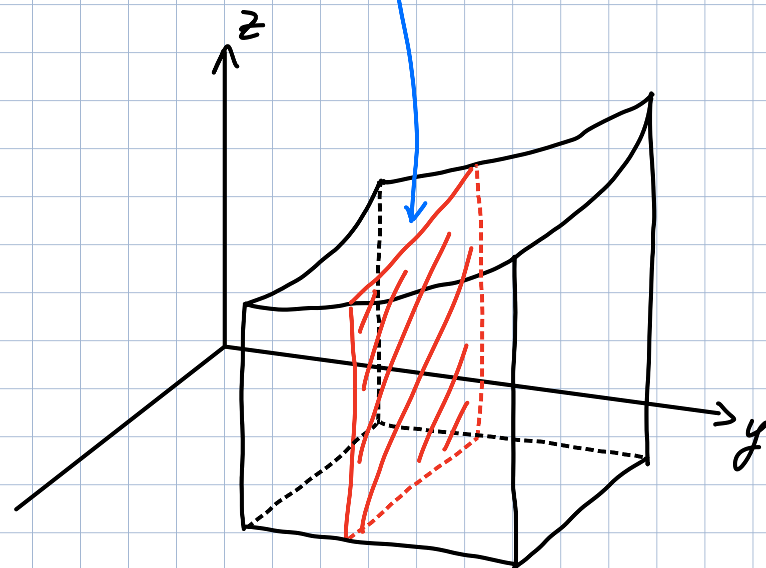
then $$\iint_R f(x,y)\,dA = \int_a^b \underbrace{\int_c^d f(x,y)\,dy}\,dx = \int_c^d \underbrace{\int_a^b f(x,y)\,dx}\,dy$$

area of the slice with fixed x

area of the slice with fixed y

Intuition:





Ex: Find $\iint_R y \sin(xy)\, dA$, where $R = [1,2] \times [0,\pi]$

Sol: $$\iint_R y\sin(xy)\,dA \underset{\text{Fubini}}{=} \int_0^{\pi} \Big(\underbrace{\int_1^2 y \sin(xy)\,dx}_{-y\frac{1}{y}\cos(xy)\big|_{x=1}^{x=2} = \cos y - \cos 2y} \Big) dy = \int_0^{\pi} (\cos y - \cos 2y)\, dy$$

$$= \sin y - \tfrac{1}{2}\sin 2y \Big|_0^{\pi} = 0$$

Rmk: We could do instead $\int_1^2 \Big(\int_0^{\pi} y \sin(xy)\, dy \Big) dx$,

but it is more complicated (need to integrate by parts twice)